b) si $\lambda = 0$ alors $\lim\limits_{+\infty} y(x) = 0$. $\boxed{\text{Montrons que } \lim\limits_{+\infty} x e^{-x^2/2} I(x) = +\infty}$,

ainsi, si $\lambda > 0$: $\lim\limits_{+\infty} y(x) = +\infty$ et si $\lambda < 0$, $\lim\limits_{+\infty} y(x) = -\infty$.

$\frac{1}{t^2} \geq \frac{t}{x^3} \Rightarrow I(x) \geq \frac{1}{x^3}\int_1^x t e^{t^2} dt = \frac{1}{2x^3}\left(e^{x^2} - e\right) \Rightarrow x e^{-x^2/2} I(x) \geq \frac{1}{x^2}\left(e^{x^2/2} - e^{-x^2/2}.e\right)$ 1

on a bien, alors, le résultat voulu.

4°/. a) $z = \frac{1}{2}(u+iv)^2 = \frac{1}{2}t^2 \Rightarrow \boxed{|z| = \frac{1}{2}|t|^2 \text{ et } \operatorname{Arg} z = 2 \operatorname{Arg} t}$ ← 2

Or : $t \in D \Leftrightarrow |t| > 0$ et $0 < \operatorname{Arg} t < \pi/2 \Rightarrow z \in D_1 \Leftrightarrow |z| > 0$ et $0 < \operatorname{Arg} z < \pi$

i.e. $x \in \mathbb{R}$ et $y > 0$. $\boxed{D_1 = \mathbb{R} \times \mathbb{R}_+^*}$ ← 3

Calcul de φ^{-1} : u vérifie : $u^4 - 2xu^2 - y^2 = 0 \Rightarrow \begin{cases} u = \sqrt{x + \sqrt{x^2+y^2}} & (u>0) \\ v = \dfrac{y}{\sqrt{x+\sqrt{x^2+y^2}}} = \sqrt{\sqrt{x^2+y^2} - x} \end{cases}$

φ^{-1} est bien définie sur D_1, donc, et

φ est une bijection de D sur D_1 ← 4

(c'est même un difféomorphisme).

b). On a : $\frac{\partial x}{\partial u} = u$, $\frac{\partial y}{\partial u} = v$, $\frac{\partial x}{\partial v} = -v$, $\frac{\partial y}{\partial v} = u$

$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}.u + \frac{\partial f}{\partial y} v$; $\frac{\partial^2 F}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u}\right) = \frac{\partial f}{\partial x} + u \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) + v \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)$.

$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} . \frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u} = \frac{\partial^2 f}{\partial x^2} . u + \frac{\partial^2 f}{\partial x \partial y} v$ (de même pour $\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)$).

$\Rightarrow \frac{\partial^2 F}{\partial u^2} = \frac{\partial f}{\partial x} + u^2 \frac{\partial^2 f}{\partial x^2} + 2uv \frac{\partial^2 f}{\partial x \partial y} + v^2 \frac{\partial^2 f}{\partial y^2}$, de même, on trouve :

$\frac{\partial^2 F}{\partial v^2} = -\frac{\partial f}{\partial x} + v^2 \frac{\partial^2 f}{\partial x^2} - 2uv \frac{\partial^2 f}{\partial x \partial y} + u^2 \frac{\partial^2 f}{\partial y^2}$ ← 6

$\Rightarrow \boxed{\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} = (u^2+v^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = (u^2+v^2) F}$ (2)

c) Si $F(u,v) = A(u) B(v)$ alors : $\frac{\partial^2 F}{\partial u^2} = A''(u).B(v)$, d'où :

(2) $\Leftrightarrow \boxed{A''(u).B(v) + A(u).B''(v) = (u^2+v^2)\, A(u).B(v)}$ ← 2 et en divisant par

$A(u).B(v)$. (2) $\Leftrightarrow \frac{A''(u)}{A(u)} - u^2 = -\left(\frac{B''(v)}{B(v)} - v^2\right) = C$ (constante ne dépendant ni de u, ni de v).

Si on prend $C = -(2n+1)$ alors on aura les équations : ← 5

$A''(u) + (2n+1-u^2)A(u) = 0$ et $B''(u) - (2n+1+u^2)B(u) = 0$.

et donc, on obtient des solutions de (2) : $A(u) = e^{-u^2/2} H_n(u)$

$B(v) = e^{+v^2/2} K_n(v)$, i.e. :

$\boxed{f(x,y) = \lambda e^{-x} H_n\left(\sqrt{x+\sqrt{x^2+y^2}}\right).K_n\left(\sqrt{\sqrt{x^2+y^2}-x}\right)}$ $\lambda \in \mathbb{R}$

en prenant $C = 2n+1$, on aurait :

$\boxed{f(x,y) = \mu e^{x} H_n\left(\sqrt{\sqrt{x^2+y^2}-x}\right).K_n\left(\sqrt{\sqrt{x^2+y^2}+x}\right)}$ $\mu \in \mathbb{R}$.