1°/ a). $T1_{[a,b]}(t) = \frac{1}{t}(e^{-ta}-e^{-tb})$, $T1_{[c,d]}(t) = \frac{1}{t}(e^{-tc}-e^{-td})$ (si $t\neq 0$).

$T(1_{[a,b]} * 1_{[c,d]})(t) = \int_{a+c}^{a+d}[x-(a+c)]e^{-tx}dx + \int_{a+d}^{b+c}(d-c)e^{-tx}dx + \int_{b+c}^{b+d}(b+d-x)e^{-tx}dx = \frac{1}{t^2}(e^{-ta}-e^{-tb})(e^{-tc}-e^{-td})$

On a donc : $T(1_{[a,b]} * 1_{[c,d]}) = T1_{[a,b]} \times T1_{[c,d]}$.

b). $f = \sum_{i=1}^{n} \lambda_i 1_{A_i}$; $g = \sum_{j=1}^{p} \mu_j 1_{B_j}$ alors :

$T(f*g) = T\left(\sum_{i,j} \lambda_i \mu_j 1_{A_i} * 1_{B_j}\right) = \sum_{i,j} \lambda_i \mu_j T(1_{A_i} * 1_{B_j}) = \sum_{i,j} \lambda_i \mu_j T1_{A_i} \times T1_{B_j}$

$= \left(\sum_{i=1}^{n} \lambda_i T1_{A_i}\right)\left(\sum_{j=1}^{p} \mu_j T1_{B_j}\right) = Tf . Tg$.

2°/ a). $m_n = -(Tf^{*n})'(0)$ or $Tf^{*n} = (Tf)^n \Rightarrow (Tf^{*n})' = n(Tf)^{n-1}(Tf)'$

et comme par hypothèse : $Tf(0) = 1$, on obtient : $m_n = -\left[n \int_{-\infty}^{+\infty} -x f(x)\,dx\right] = n m_1$

$\sigma_n = (Tf^{*n})''(0) - m_n^2 = n(n-1)\underbrace{[(Tf)'(0)]^2}_{=m_1^2} + n(Tf)''(0) - m_n^2 = n(n-1).m_1^2 + n(\sigma_1 + m_1^2) - n^2 m_1^2$

$= n\sigma_1$.

b). $Tg_n(t) = \int_{-\infty}^{+\infty} e^{-tx} f^{*n}(nx)\, n\,dx = \int_{-\infty}^{+\infty} e^{-\frac{t}{n}u} f^{*n}(u)\,du = (Tf^{*n})\left(\frac{t}{n}\right) = (Tf)^n\left(\frac{t}{n}\right)$.

(en posant $u = nx$)

T.Y : $Tf\left(\frac{t}{n}\right) = Tf(0) + \frac{t}{n}(Tf)'(0) + o\left(\frac{t}{n}\right)$ mais $Tf(0) = 1$ donc, la limite de : $n \log Tf\left(\frac{t}{n}\right)$ est égale à $t(Tf)'(0)$ d'où $\lim_{n\to+\infty} Tg_n(t) = e^{t(Tf)'(0)}$.

c). $Th_n(t) = (Tf)^n\left(\frac{t}{\sqrt{n}}\right)$.

T.Y. : $Tf\left(\frac{t}{\sqrt{n}}\right) = Tf(0) + \frac{t}{\sqrt{n}}(Tf)'(0) + \frac{t^2}{2n}(Tf)''(0) + o\left(\frac{t^2}{n}\right)$ Or : $Tf(0) = 1$ et $(Tf)'(0) = 0$.

donc, $\lim_{n\to+\infty} Th_n(t) = e^{t^2(Tf)''(0)}$.

Exercices :

I. f : on pose : $t = \sqrt{1-x^3}$ $\Rightarrow \int f(x)\,dx = -\frac{2}{3}\int \frac{dt}{1+t^2} = -\frac{2}{3}\mathrm{Arctg}\left(\sqrt{1-x^3}\right) + C$

g : on pose $u = \mathrm{tg}\,x \Rightarrow \int g(x)\,dx = I = \int \frac{du}{(1+u^2)\sqrt{2+u^2}} \underset{u=t\sqrt{2}}{=} \int \frac{dt}{(1+2t^2)\sqrt{1+t^2}} \underset{\mathrm{sh}\,v = t}{=} \int \frac{dv}{1+2\,\mathrm{sh}^2 v}$

Or $1 + 2\,\mathrm{sh}^2 v = 1 + \frac{2\,\mathrm{th}^2 v}{1-\mathrm{th}^2 v} = \frac{1+\mathrm{th}^2 v}{1-\mathrm{th}^2 v} \Rightarrow I = \int \frac{(1-\mathrm{th}^2 v)\,dv}{1+\mathrm{th}^2 v} = \mathrm{Arctg}(\mathrm{th}\,v) + C$

et $\mathrm{th}\,v = \frac{t}{\sqrt{1+t^2}} = \frac{u}{\sqrt{2+u^2}} \Rightarrow \int g(x)\,dx = \mathrm{Arctg}\frac{\mathrm{tg}\,x}{\sqrt{2+\mathrm{tg}^2 x}} + C = \varepsilon\,\mathrm{Arcsin}\left(\frac{\sin x}{\sqrt{2}}\right)$.

ε du signe de $\cos x$.

II. On a : $\frac{1}{n+p+1} < \frac{1}{\sqrt{(n+p)(n+p+1)}} < \frac{1}{n+p} \Rightarrow v_n < u_n < w_n$

où $v_n = \frac{1}{n}\sum_{p=1}^{n-1} \frac{1}{1+\frac{p+1}{n}}$ et $w_n = \frac{1}{n}\sum_{p=1}^{n} \frac{1}{1+\frac{p}{n}}$ Or : $\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} w_n = \int_1^2 \frac{dx}{x} = \log 2$

donc : $\lim_{n\to+\infty} u_n = \log 2$