Exercice I.

1°/. a). $f(h) - f(0) = h f'(\theta h) \Rightarrow \theta = 4/9 = 0,444$

b). $f(1+1) - f(1) = f'(1+\theta h) \Rightarrow \theta = 3 - \frac{16\sqrt{2}}{9} = 0,486$.

2°/. $\theta = \frac{1}{h}\left[\left(\frac{2\left[(1+h)^{3/2} - 1\right]}{3h}\right)^2 - 1\right]$. $h \geq -1$

$$\theta = \frac{1}{h}\underbrace{\left(\frac{2\left([1+h]^{3/2} - 1\right)}{3h} - 1\right)}_{A(h)}\underbrace{\left(\frac{2\left([1+h]^{3/2} - 1\right)}{3h} + 1\right)}_{B(h)}.$$

$A(h) = \frac{2}{3h}\,\frac{(1+h)^3 - 1}{(1+h)^{3/2} + 1}$: $\lim_{h\to 0} A(h) = 1 \Rightarrow \lim_{h \to 0} B(h) = 2$

$A(h) - 1 = \frac{2}{3h}\left[(1+h)^{3/2} - (1 + 3/2\, h)\right] = \frac{2}{3h}\,\frac{3/4\, h^2 + h^3}{(1+h)^{3/2} + (1 + \frac{3}{2}h)}$

$\Rightarrow \lim_{h\to 0} \frac{1}{h}(A(h) - 1) = \frac{1}{4}$ et $\theta(h) = \frac{1}{h}(A(h) - 1) . B(h) \xrightarrow[h\to 0]{} 1/2$.

3°/. $x \longmapsto |x|^{3/2}$ continue sur $\mathbb{R}$, dérivable sur $\mathbb{R}$ (même en 0).

$\Rightarrow f(1-3) - f(1) = -3 f'(1 + \theta.(-3))$. $\Rightarrow f'(1 - 3\theta) = -\frac{2\sqrt{2} - 1}{3} < 0$

donc, $1 - 3\theta < 0 \Rightarrow f'(1 - 3\theta) = -\frac{3}{2}(3\theta - 1)^{1/2}$

d'où : $\theta = \frac{1}{3}\left[1 + \left(\frac{2(2\sqrt{2} - 1)}{9}\right)^2\right] = 0,388$

Problème :

I - 1°/. $g(x)$ est définie, continue et dérivable sur $\mathbb{R} - \{1\}$.

$g'(x) = \frac{1}{x^2 + 4x + 5} - \frac{1}{1 + x^2} = \frac{-4(x+1)}{(1+x^2)(x^2 + 4x + 5)}$.

x	$-\infty$		-1		$+1$		$+\infty$
$g'(x)$		$+$	0	$-$	||	$-$	
$g(x)$	$\frac{\pi}{2} - \mathrm{Arctg}\,3$ positif	$\nearrow$	$\mathrm{Arctg}\,2 + \frac{\pi}{4}$	$\searrow$	$\frac{\pi}{4}$ || $-\frac{3\pi}{4}$	$\searrow$	$-\frac{\pi}{2} - \mathrm{Arctg}\,3$

2°/. Discussion :

$m > \mathrm{Arctg}\,2 + \frac{\pi}{4}$: $S = \emptyset$.

$m =$ " : $S = \{-1\}$.

$\frac{\pi}{4} < m < \mathrm{Arctg}\,2 + \frac{\pi}{4}$: $S = \{x_1, x_2\}$.

$\frac{\pi}{2} - \mathrm{Arctg}\,3 < m \leq \frac{\pi}{4}$: $S = \{x_1\}$. $x_1 < -1$.

$-\frac{3\pi}{4} \leq m \leq \frac{\pi}{2} - \mathrm{Arctg}\,3$: $S = \emptyset$.

$-\frac{\pi}{2} - \mathrm{Arctg}\,3 < m < -\frac{3\pi}{4}$: $S = \{x_1\}$ $x_1 > 1$.

$m \leq -\frac{\pi}{2} - \mathrm{Arctg}\,3$: $S = \emptyset$.

3°/. a) $m = \frac{\pi}{4}$: 1 sol. $x_1 < -1$:

$g(x) = \frac{\pi}{4} \Leftrightarrow \mathrm{Arctg}\,\frac{3x+7}{1-x} = \mathrm{Arctg}\,x + \mathrm{Arctg}\,1 = \mathrm{Arctg}\,\frac{1+x}{1-x}$ (formule applicable).

$\Leftrightarrow 3x + 7 = 1 + x \Leftrightarrow x = -3$

b). 2 solutions : $x_1 = -1 - \sqrt{\frac{2}{3}}$; $x_2 = -1 + \sqrt{\frac{2}{3}}$.

c) $-\frac{3\pi}{4} < -\frac{5\pi}{12} < \frac{\pi}{2} - \mathrm{Arctg}\,3$: $S = \emptyset$.

d) 1 seule solution > 1 : $x = -1 + \sqrt{14}$.

4°/. $g(x) = m$ a 2 sol. < -1 : on peut écrire $g(x) = m \Leftrightarrow \mathrm{Arctg}\,\frac{3x+7}{1-x} = \mathrm{Arctg}\,x + m = \mathrm{Arctg}\,\frac{x+k}{1-kx}$ ($k = \mathrm{tg}\,m$) ($kx < 1$)

$\Leftrightarrow x^2(3k-1) + 2(3k-1)x + k - 7 = 0 \Rightarrow \boxed{x_1 + x_2 = -2}$