I - 1°).

a). $f_2 = \frac{1}{2\pi}\int_0^{2\pi}\cos^2 t\, e^{x\cos t}\,dt = f_0 - \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos t}\sin^2 t\,dt \Rightarrow x f_2 + f_1 - x f_0 = 0$

b) $f_n = f_{n-2} - \frac{1}{2\pi}\int_0^{2\pi}(\cos t)^{n-2}\sin^2 t\, e^{x\cos t}\,dt \Rightarrow \boxed{x f_{n+1} + n f_n - x f_{n-1} - (n-1) f_{n-2} = 0}$

$n f_n(0) = (n-1) f_{n-2}(0) \Rightarrow f_n(0) = \frac{n-1}{n} f_{n-2}(0)$.

$f_0(0) = \frac{1}{2\pi}\int_0^{2\pi} dt = 1$; $f_1(0) = \frac{1}{2\pi}\int_0^{2\pi}\cos t\,dt = 0 \Rightarrow$

$n=2p$: $f_{2p}(0) = \frac{2p-1}{2p}\cdot\frac{2p-3}{2p-2}\cdots\times\frac{1}{2} = \frac{(2p)!}{2^{2p}(p!)^2}$; $n=2p+1$: $f_{2p+1}(0)=0$.

2°) a). $e^{x\cos t} = e^{x_0\cos t} + (x-x_0)\cos t + \frac{(x-x_0)^2}{2}\cos^2 t\, e^{c\cos t}$

$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2} f''(c)$ $\quad c\in[x, x_0]$.

$\varepsilon(x,t) = \frac{x-x_0}{2}\cos^2 t\, e^{c\cos t} \Rightarrow |\varepsilon(x,t)| \le \frac{|x-x_0|}{2}\, e^{|c|}$ et $|c| \le |x|+|x_0|$

$\Rightarrow$ l'inégalité demandée.

b). $f_0(x) - f_0(x_0) = \frac{1}{2\pi}\int_0^{2\pi}\left(e^{x\cos t} - e^{x_0\cos t}\right)dt = \frac{1}{2\pi}(x-x_0)\int_0 \cos t\, e^{x\cos t}\,dt + \frac{1}{2\pi}(x-x_0)\int_0 \varepsilon(x,t)\,dt.$

$\Rightarrow \left|\frac{f_0(x)-f_0(x_0)}{x-x_0} - \frac{1}{2\pi}\int_0^{2\pi}\cos t\, e^{x\cos t}\,dt\right| \le \frac{1}{2\pi}\left|\int_0^{2\pi}|\varepsilon(x,t)|\,dt\right| \le \frac{1}{\cancel{2\pi}}\frac{|x-x_0|}{2}\, e^{|x|+|x_0|}$.

$\Rightarrow f_0'(x) = f_1(x)$.

c). f_0 indéfiniment dérivable : par récurrence :

$f_0^{(n)}(x) = f_n(x)$. $\quad f_n(x) - f_n(x_0)$ etc ... (ça marche bien)

D.L. : $f_0(x) = \frac{x^2}{2}\times\frac{2!}{2^2\,1!^2} + \frac{x^4}{4!}\cdot\frac{4!}{2^4(2!)^2} + \dots + \frac{x^{2n}}{(2n)!}\,\frac{(2n)!}{2^{2n}(n!)^2} + o(x^{2n})$.

$$\boxed{f_0(x) = \frac{x^2}{4} + \frac{x^4}{64} + \dots + \frac{x^{2n}}{2^{2n}(n!)^2} + o(x^{2n})}$$

3°) a). On a : $f_n(x) = \frac{1}{\pi}\int_0^{\pi}(\cos t)^n e^{x\cos t}\,dt$. $\quad t\to\pi-t \Rightarrow f_n(x) = \frac{2}{\pi}\int_0^{\pi/2}(\cos t)^n\,\frac{e^{x\cos t}+(-1)^n e^{-x\cos t}}{2}\,dt$.

$n=2p$: $f_{2p}(x) = \frac{2}{\pi}\int_0^{\pi/2}(\cos t)^{2p}\operatorname{ch}(x\cos t)\,dt$. $\quad n=2p+1$: $f_{2p+1}(x) = \frac{2}{\pi}\int_0^{\pi/2}(\cos t)^{2p+1}\operatorname{sh}(x\cos t)\,dt$.

b). $f_0(x) \ge \frac{2}{\pi}\int_0^{\pi/3}\operatorname{ch}(x\cos t)\,dt \ge \frac{2}{3}\operatorname{ch}\frac{x}{2}$ de même : $f_1(x)\ge\frac{1}{3}\operatorname{sh}\frac{x}{2}$.

or $0\le t\le\pi/3$: $\cos t\ge 1/2$

c). f_0 : paire sur $[0,+\infty[$: $f_0' = f_1 \ge \frac{1}{3}\operatorname{sh}\frac{x}{2} \ge 0$.

x	0	$+\infty$
f_1	$+$	
f_0	0 ↗	$+\infty$