II. $F(x) = \frac{x+7}{2}\sqrt{x^2-2x+4} + 2\,\mathrm{Argsh}\,\frac{x-1}{2} + C = \frac{x+7}{2}\sqrt{x^2-2x+4} + 2\log\left|(x-1)+\sqrt{x^2-2x+5}\right| + C$ (2)

$G(x) = -\frac{1}{4\cos x} - \frac{1}{8}\log\frac{1+\cos x}{1-\cos x} + C = -\frac{1}{4\cos x} + \frac{1}{4}\log\operatorname{tg}\frac{x}{2} + C.$

$H(x) = 2x\left[1+\sqrt{1+e^x}\right] - 4\sqrt{1+e^x} - 4\log\left[1+\sqrt{1+e^x}\right] + C.$

intégration par parties + $t = \sqrt{1+e^x}$

$$= 2x\sqrt{1+e^x} - 4\sqrt{1+e^x} + 2\log\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} + C.$$

III. 1°) a) $a_{p,0} = \frac{1}{p+1}$; $a_{0,n} = \frac{1}{n+1}$; $a_{p,1} = \int_0^1 x^p(1-x)\,dx = a_{p,0} - a_{p+1,0} = \frac{1}{p+1} - \frac{1}{p+2} = \frac{1}{(p+1)(p+2)}$

$a_{1,n} = \frac{1}{(n+1)(n+2)}.$

b). $a_{p,n} = \int_0^1 x^p(1-x)^n\,dx$ (par parties). $\boxed{a_{p,n} = \frac{p}{n+1}\,a_{p-1,n+1}}$

$(1-x)^n dx = dv \Rightarrow v = -\frac{1}{n+1}(1-x)^{n+1}$ et de m̂ : $p \geq 1$

$x^p = u \qquad du = p\,x^{p-1}$

$x^p\,dx = dv \Rightarrow v = \frac{1}{p}\,x^{p-1}.$

$(1-x)^n = u \Rightarrow du = -n(1-x)^n dx.$ $\Big\} \Rightarrow a_{p,n} = \frac{n}{p+1}\,a_{p+1,n-1}$

c). $a_{p,n} = \frac{p(p-1)(\dots)1}{(n+1)(n+2)(\dots)(n+p)}\,a_{0,n+p} = \frac{p!\times n!}{(n+p)!}\times\frac{1}{n+p+1} = \frac{1}{(n+p+1)\,C_{n+p}^{p}}.$

$n\to+\infty$: $a_{p,n} \sim \frac{p!}{n^{p+1}}$ évident.

2°) a). $|I_n| \leq \int_0^1 (1-x)^n |f(x)|\,dx \leq \|f\|.\int_0^1 (1-x)^n dx = \frac{1}{n+1}\,\|f\|.$

$f(x) = f(0) + x.f'(\theta x) \Rightarrow I_n = \int_0^1 (1-x)^n.f(0)\,dx + \int_0^1 (1-x)^n x.f'(\theta x)\,dx.$

$\frac{f(0)}{n+1}$ +

$\Rightarrow \left|I_n - \frac{f(0)}{n+1}\right| \leq \|f'\|\times\frac{1}{(n+1)(n+2)}$ $\Rightarrow$ $I_n = \frac{f(0)}{n}(1+\varepsilon_n)$. si $\boxed{f(0)\neq 0}$

b). $f(x) = f(0) + x f'(0) + \dots + \frac{x^p}{p!}f^{(p)}(0) + \frac{x^{p+1}}{(p+1)!}f^{(p)}(\theta.x).$

$\Rightarrow I_n = \frac{1}{n+1}f(0) + \frac{1}{(n+1)(n+2)}f'(0) + \dots + \frac{n!}{(n+p)!}f^{(p)}(0) + \cancel{x}.$

$\Rightarrow \left|I_n - (\quad\dots\quad)\right| \leq \|f^{(p)}\|.a_{p+1,n}.$ et $a_{p+1,n} \sim \frac{(p+1)!}{n^{p+2}}$

$\Rightarrow n^{p+1}a_{p+1,n} \xrightarrow[n\to+\infty]{} 0.$

c). $I_n = \frac{f(0)}{n+1} + \frac{f'(0)}{(n+1)(n+2)} + \frac{f''(0)}{(n+1)(n+2)(n+3)} + o\left(\frac{1}{n^3}\right).$